What Can We Measure?

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Abstract

In this chapter we approach the question of "what is measurable" from an abstract point of view using ideas from geometric measure theory. As it turns out such a first principles approach gives us quantities such as mean and Gaussian curvature integrals in the discrete setting and more generally, fully characterizes a certain class of possible measures. Consequently one can characterize all possible "sensible" measurements in the discrete setting which may form the basis for physical simulation, for example.

1 Introduction

When characterizing a shape or changes in shape we must first ask, what can we measure about a shape? For example, for a region in $\mathbb{R}^3$ we may ask for its volume or its surface area. If the object at hand undergoes deformation due to forces acting on it we may need to formulate the laws governing the change in shape in terms of measurable quantities and their change over time. Usually such measurable quantities for a shape are defined with the help of integral calculus and often require some amount of smoothness on the object to be well defined. In this chapter we will take a more abstract approach to the question of measurable quantities which will allow us to define notions such as mean curvature integrals and the curvature tensor for piecewise linear meshes without having to worry about the meaning of second derivatives in settings in which they do not exist. In fact in this chapter we will give an account of a classical result due to Hadwiger [Hadwiger 1957], which shows that for a convex, compact set in $\mathbb{R}^n$ there are only $n+1$ independent measurements if we require that the measurements be invariant under Euclidean motions (and satisfy certain "sanity" conditions). We will see how these measurements are constructed in a very straightforward and elementary manner and that they can be read off from a characteristic polynomial due to Steiner [Steiner 1840]. This polynomial describes the volume of a family of shapes which arise when we "grow" a given shape. As a practical tool arising from these consideration we will see that there is a well defined notion of the curvature tensor for piecewise linear meshes and we will see very simple formulæfor quantities needed in physical simulation with piecewise linear meshes. Much of the treatment here will be limited to convex bodies to keep things simple.

The treatment in this chapter draws heavily upon work by Gian-Carlo Rota and Daniel Klein, Hadwigers pioneering work, and some recent work by David Cohen-Steiner and colleagues.

2 Geometric Measures

To begin with let us define what we mean by a measure. A measure is a function μ defined on a family of subsets of some set S, and it takes on real values: $\mu : L \rightarrow \mathbb{R}$. Here L denotes this family of subsets and we require of L that it is closed under finite set union and intersection as well as that it contains the empty set, $\emptyset \in L$. The measure μ must satisfy two axioms: (1) $\mu(\emptyset) = 0$; and (2) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ whenever A and B are measurable. The first axiom is required to get anything that has a hope of being well defined. If $\mu(\emptyset)$ was not equal to zero the measure of some set $\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$ could not be defined. The second axiom captures the idea that the measure of the union of two sets should be the sum of the measures minus the measure of their overlap. For example, consider the volume of the union of two sets which clearly has this property. It will also turn out that the additivity property is the key to reducing measurements for complicated sets to measurements on simple sets. We will furthermore require that all measures we consider be invariant under Euclidean motions, *i.e.*, translations and rotations. This is so that our measurements do not depend on where we place the coordinate origin and how we orient the coordinate axes. A measure which depended on these wouldn't be very useful.

Let's see some examples. A well known example of such a measure is the volume of bodies in $\mathbb{R}^3$. Clearly the volume of the empty body is zero and the volume satisfies the additivity axiom. The volume also does not depend on where the coordinate origin is placed and how the coordinate frame is rotated. To uniquely tie down the volume there is one final ambiguity due to the units of measurement being used, which we must remove. To this end we enforce a normalization which states that the volume of the unit, coordinate axis-aligned parallelepiped in $\mathbb{R}^n$ be one. With this we get

$$\mu_n^n(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n$$

for x_1 to x_n the side lengths of a given axis-aligned parallelepiped. The superscript n denotes this as a measure on $\mathbb{R}^n$, while the subscript denotes the type of measurement being taken. Consider now a translation of such a parallelepiped. Since such a transformation does not change the side lengths μ^n is translation invariant. The same applies to a rotation of such a parallelepiped. Overall we say that this measure is *rigid motion invariant*. Notice that we have only defined the meaning of μ_n^n for axis-aligned parallelepipeds as well as finite unions and intersections of such parallelepipeds. The definition can be extended to more general bodies through a limiting process akin to how Riemann integration fills the domain with ever smaller boxes to approach the entire domain in the limit. There is nothing here that prevents us from performing the same limit process. In fact we will see later that once we add this final requirement, that the measure is continuous in the limit, the class of such measures is completely tied down. This is Hadwiger's famous theorem. But, more on that later.

Of course the next question is, are there other such invariant measures? Here is a proposal:

$$\mu^n_{n-1}(x_1,\ldots,x_n) = \\ x_1x_2 + x_1x_3 + \ldots + x_1x_n + x_2x_3 + \ldots + x_2x_n \ldots$$

For an axis-aligned parallelepiped in $\mathbb{R}^3$ we'd get

$$\mu^3_2(x_1,x_2,x_3) = x_1x_2 + x_2x_3 + x_3x_1$$

which is just half the surface area of the parallelepiped with sides x_1, x_2, and x_3. Since we have the additivity property we can certainly extend this definition to more general bodies through a limiting process and find that we get, up to normalization, the surface area.

Continuing in this fashion we are next led to consider

$$\mu^3_1(x_1,x_2,x_3) = x_1 + x_2 + x_3$$

(and similarly for μ^n_1). For a parallelepiped this function measures one quarter the sum of lengths of its bounding edges. Once again this new measure is rigid motion invariant since the side lengths x_1, x_2 and x_3 do not change under a rigid motion. What we need to check is whether it satisfies the additivity theorem. Indeed it does, which is easily checked for the union of two axis-aligned parallelepipeds if the result is another axis-aligned parallelepiped. What is less clear is what this measure represents if we extend it to more general shapes where the notion of "sum of edge lengths" is not clear. The resulting continuous measure is sometimes referred to as the *mean width*.

From these simple examples we can see a pattern. For Euclidean n-space we can use the elementary symmetric polynomials in edge lengths to define n invariant measures

$$\mu^n_k(x_1,\ldots,x_n) = \sum_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$$

for $k = 1,\ldots,n$ for parallelepipeds. To extend this definition to more general bodies as alluded to above, we'll follow ideas from geometric probability. In particular we will extend these measures to the ring of compact convex bodies, *i.e.*, finite unions and intersections of compact convex sets in $\mathbb{R}^n$.

3 How Many Points, Lines, Planes,... Hit a Body?

Consider a compact convex set, a *convex body*, in $\mathbb{R}^n$ and surround it by a box. One way to measure its volume is to count the number of points that, when randomly thrown into the box, hit the body versus those that hit empty space inside the box. To generalize this idea we consider *affine subspaces* of dimension $k < n$ in $\mathbb{R}^n$. Recall that an affine subspace of dimension k is spanned by $k+1$ points $p_i \in \mathbb{R}^n$ (in general position), *i.e.*, the space consists of all points q which can be written as affine combinations $q = \sum_i \alpha_i p_i$, with $\sum_i \alpha_i = 1$. Such an affine subspace is simply a linear subspace translated, *i.e.*, it does not necessarily go through the origin. For example, for $k = 1$, $n = 3$ we will consider all lines—a line being the set of points one can generate as affine combinations of two points on the line—in three space. Let's denote the measure of all lines going through a rectangle R in $\mathbb{R}^3$ as $\lambda^3_1(R)$. The claim is that

$$\lambda^3_1(R) = c\mu^3_2(R),$$

i.e., the measure of all lines which meet the rectangle is proportional to the area of the rectangle. To see this, note that a given line (in general position) either meets the rectangle once or not at all. Conversely for a given point in the rectangle there is a whole set of lines—a sphere's worth—which "pierce" the rectangle in the given point. The measure of those lines is proportional to the area of the unit sphere. Since this is true for all points in the rectangle we see that the total measure of all such lines must be proportional to the area of the rectangle with a constant of proportionality depending on the measure of the sphere. For now such constants are irrelevant for our considerations so we will just set them to unity. Given a more complicated shape C in a plane nothing prevents us from performing a limiting process and we see that the measure of lines meeting C is

$$\lambda^3_1(C) = \mu^3_2(C),$$

i.e., it is proportional to the area of the region C. Given a union of rectangles $D = \cup_i R_i$, each living in a different plane, we get

$$\int X_D(\omega)\,d\lambda^3_1(\omega) = \sum_i \mu^3_2(R_i).$$

Here $X_D(\omega)$ counts the number of times a line ω meets the set D and the integration is performed over all lines. Going to the limit we find for any convex body E a measure proportional to its surface area

$$\int X_E(\omega)\,d\lambda^3_1(\omega) = \mu^3_2(E).$$

Using planes ($k = 2$) we can now generalize the mean width. For a straight line segment $c \in \mathbb{R}^3$ we find $\lambda^3_2(c) = \mu^3_1(c) = l(c)$, *i.e.*, the measure of all planes that meet the straight line segment is proportional—as before we set the constant of proportionality to unity—to the length of the line segment. The argument mimics what we said above: a plane either meets the line once or not at all. For a given point on the line there is once again a whole set of planes going through that point. Considering the normals to such planes we see that this set of planes is proportional in measure to the unit sphere without being more precise about the actual constant of proportionality. Once again this can be generalized with a limiting process giving us the measure of all planes hitting an arbitrary curve in space as proportional to its length

$$\int X_F(\omega)\,d\lambda^3_2(\omega) = \mu^3_1(F).$$

Here the integration is performed over all planes $\omega \in \mathbb{R}^3$, and X_F counts the number of times a given plane touches the curve F.

It is easy to see that this way of measuring recovers the perimeter of a parallelepiped as we had defined it before

$$\lambda^3_2(P) = \mu^3_1(P).$$

To see this consider the integration over all planes but taken in groups. With the parallelepiped having one corner at the origin—and being axis-aligned—first consider all planes whose normal (n_x, n_y, n_z) has all non-negative entries (*i.e.*, the normal points into the positive octant). Now consider a sequence of three edges of P, connected at their endpoints, going from one corner to its opposing corner. For example, first traversing an edge parallel to the first coordinate axis, then an edge parallel to the second coordinate axis and finally an edge parallel to the third coordinate axis. The total length of this curve will be the sum of lengths of the three segments (x_1, x_2 and x_3 respectively). Given a plane with normal pointing into the positive octant and meeting the parallelepiped P we see it must meet our sequence of three edges in exactly one point. From this it follows that the measure of all such planes is given by the length of the sequence of edges $\mu_1^3(P) = x_1 + x_2 + x_3$ (up to a constant of proportionality). The same argument holds for the remaining seven octants giving us the desired result up to a constant. We can now see that $\mu_1^3(E)$ for some convex body E can be written as

$$\int X_E(\omega)\, d\lambda_2^3(\omega) = \mu_1^3(E),$$

i.e., the measure of all planes which meet E. With this we have generalized the notion of perimeter to more general sets.

We have now seen n different Euclidean motion invariant measures $\mu_k^n(C)$, given as the measure of all affine subspaces of dimension $n-k$ meeting $C \subset \mathbb{R}^n$ for $k = 1, \dots, n$. These measures are called the *intrinsic volumes*. Clearly any linear combination of these measures is also rigid motion invariant. It is natural to wonder then whether these linear combinations generate all such measures. It turns out there is one final measure missing in our basis set of measures before we arrive at Hadwiger's theorem. This measure corresponds to the elementary symmetric polynomial of order zero

$$\mu_0(x_1, \dots, x_n) = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \end{cases}$$

This very special measure will turn out to be the Euler characteristic of a convex body which takes on the value 1 on all non-empty compact convex bodies. To show that everything works as advertised we use induction on the dimension. In dimension $n = 1$ we consider closed intervals $[a, b]$, $a < b$. Instead of working with the set directly we consider a functional on the characteristic function $f_{[a,b]}$ of the set which does the trick

$$\chi_1(f) = \int_{\mathbb{R}} f(\omega) - f(\omega+)\, d\omega.$$

Here $f(\omega+)$ denotes the right limiting value of f at ω: $\lim_{\epsilon \to 0} f(\omega + \epsilon)$, $\epsilon > 0$. For the set $[a, b]$, $f(\omega) - f(\omega+)$ is zero for all $\omega \in \mathbb{R}$ except b since $f(b) = 1$ and $f(b+) = 0$. Now we use induction to deal with higher dimensions. In $\mathbb{R}^n$ take a straight line L and consider the affine subspaces A_ω of dimension $n-1$ which are orthogonal to L and parameterized by ω along L. Letting f be the characteristic function of a convex body in $\mathbb{R}^n$ we get

$$\chi_n(f) = \int_{\mathbb{R}} \chi_{n-1}(f_\omega) - \chi_{n-1}(f_{\omega+})\, d\omega.$$

Here f_ω is the restriction of f to the affine space A_ω or alternatively the characteristic function of the intersection of A_ω and the convex body of interest, while $f_{\omega+}$ is defined as before as the limit of f_ω from above. With this we define $\mu_0^n(G) = \chi_n(f)$ for any finite union of convex bodies G and f the characteristic function of the set $G \in \mathbb{R}^n$.

That this definition of μ_0^n amounts to the Euler characteristic is not immediately clear, but it is easy to show, if we convince ourselves that for any non empty convex body $C \in \mathbb{R}^n$

$$\mu_0^n(\mathrm{Int}(C)) = (-1)^n.$$

For $n = 1$, *i.e.*, the case of open intervals on the real line, this statement is obviously correct. We can now apply the recursive definition to the characteristic function of the interior of C and get

$$\mu_0^n(\mathrm{Int}(C)) = \int_\omega \chi_{n-1}(f_\omega) - \chi_{n-1}(f_{\omega+})\, d\omega.$$

By induction the right hand side is zero except for the first ω at which $A_\omega \cap C$ is non-empty. There $\chi_{n-1}(f_{\omega+}) = (-1)^{n-1}$, thus proving our assertion for all n.

The Euler-Poincaré formula for a convex polyhedron in $\mathbb{R}^3$

$$|F| - |E| + |V| = 2$$

which relates the number of faces, edges, and vertices to one another now follows easily. Given a convex polyhedron simply write it as the non-overlapping union of the interiors of all its cells from dimension n down to dimension 0, where the interior of a vertex (0-cell) is the vertex itself. Then

$$\mu_0^n(P) = \sum_{c \in P} \mu_0^n(\mathrm{Int}(c)) = c_0 - c_1 + c_2 - \dots$$

where c_i equals the number of cells of dimension i. For the case of a polyhedron in $\mathbb{R}^3$ this is exactly the Euler-Poincaré formula as given above since for $n = 3$ we have

$$1 = \mu_0^3(P) = c_0 - c_1 + c_2 - c_3 = |V| - |E| + |F| - 1.$$

4 The Intrinsic Volumes and Hadwiger's Theorem

The above machinery can now be used to define the intrinsic volumes as functions of the Euler characteristic alone for all finite unions G of convex bodies

$$\mu_k^n(G) = \int \mu_0^n(G \cap \omega)\, d\lambda_{n-k}^n(\omega).$$

Here $\mu_0^n(G \cap \omega)$ plays the role of $X_G(\omega)$ we used earlier to count the number of times ω hits G.

There is one final ingredient missing, continuity in the limit. Suppose C_n is a sequence of convex bodies which converges to C in the limit as $n \to \infty$. Hadwiger's theorem says that if a Euclidean motion invariant measure μ of convex bodies in $\mathbb{R}^n$ is continuous in the sense that

$$\lim_{C_n \to C} \mu(C_n) = \mu(C)$$

then μ must be a linear combination of the intrinsic volumes μ_k^n, $k = 0, \dots, n$. In other words, the intrinsic volumes, under the additional assumption of continuity, are the only linearly independent, Euclidean motion invariant, additive measures on finite unions (and intersections) of convex bodies in $\mathbb{R}^n$.

What does all of this have to do with the applications we have in mind? A consequence of Hadwiger's theorem assures us that if we want to take measurements of piecewise linear geometry (surface or volume meshes, for example) such measurements should be functions of the intrinsic volumes. This assumes of course that we are looking for additive measurements which are Euclidean motion invariant and continuous in the limit. For a triangle for example this would be area, edge length, and Euler characteristic. Similarly for a tetrahedron with its volume, surface area, mean width, and Euler characteristic. As the name suggests all of these measurements are intrinsic, *i.e.*, they can be computed without requiring an embedding. All that is needed is a metric to compute the intrinsic volumes. Of course in practice the metric is often induced by an embedding.

5 Steiner's Formula

We return now to questions of discrete differential geometry by showing that the intrinsic volumes are intricately linked to curvature integrals and represent their generalization to the non-smooth setting. This connection is established by Steiner's formula [Steiner 1840].

Consider a non-empty convex body $K \in \mathbb{R}^n$ together with its parallel bodies

$$K_\epsilon = \{x \in \mathbb{R}^n : d(x, K) \leq \epsilon\}$$

where $d(x, K)$ denotes the Euclidean distance from x to the set K. In effect K_ϵ is the body K thickened by ϵ. Steiner's formula gives the volume of K_ϵ as a polynomial in ϵ

$$V(K_\epsilon) = \sum_{j=0}^{n} V(\mathbb{B}_{n-j}) V_j(K) \epsilon^{n-j}.$$

Here the $V_j(K)$ correspond to the measures μ_k^n we have seen earlier if we let $k = n - j$. For this formula to be correct the $V_j(K)$ are normalized so that they compute the j-dimensional volume when restricted to a j-dimensional subspace of $\mathbb{R}^n$. (Recall that we ignored normalizations in the definition of the μ_k^n.) $V(\mathbb{B}_k) = \pi^{k/2}/\Gamma(1 + k/2)$ denotes the k-volume of the k-unit ball. In particular we have $V(\mathbb{B}_0) = 1$, $V(\mathbb{B}_1) = 2$, $V(\mathbb{B}_2) = \pi$, and $V(\mathbb{B}_3) = 4\pi/3$.

In the case of a polyhedron we can verify Steiner's formula "by hand." Consider a tetrahedron in $T \in \mathbb{R}^3$ and the volume of its parallel bodies T_ϵ. For $\epsilon = 0$ we have the volume of T itself ($V_3(T)$). The first order term in ϵ, $2V_2(T)$, is controlled by area measures: above each triangle a displacement along the normal creates additional volume proportional to ϵ and the area of the triangle. The second order term in ϵ, $\pi V_1(T)$, corresponds to edge lengths and dihedral angles. Above each edge the parallel bodies form a wedge with radius ϵ and opening angle θ, which is the exterior angle of the faces meeting at that edge. and the length of the edge. The volume of each such wedge is proportional to edge length, exterior angle, and ϵ^2. Finally the third order term in ϵ, $4\pi/3V_0(T)$, corresponds to the volume of the parallel bodies formed over vertices. Each vertex gives rise to additional volume spanned by the vertex and a spherical cap above it. The spherical cap corresponds to a spherical triangle formed by the three incident triangle normals. The volume of such a spherical wedge is proportional to its solid angle and ϵ^3.

If we have a convex body with a boundary which is C^2 we can give a different representation of Steiner's formula. Consider such a convex $M \in \mathbb{R}^n$ and define the offset function

$$g(p) = p + t\vec{n}(p)$$

for $0 \leq t \leq \epsilon$, $p \in \partial M$ and $\vec{n}(p)$ the outward normal to M at p. We can now directly compute the volume of M_ϵ as the sum of $V_n(M)$ and the volume between the surfaces ∂M and ∂M_ϵ. The latter can be written as an integral of the determinant of the Jacobian of g

$$\int_{\partial M} \left(\int_0^\epsilon \left| \frac{\partial g(p)}{\partial p} \right| dt \right) dp.$$

Since we have a choice of coordinate frame in which to do this integration we may assume without loss of generality that we use principal curvature coordinates on ∂M, *i.e.*, a set of orthogonal directions in which the curvature tensor diagonalizes. In that case

$$\begin{aligned} \left| \frac{\partial g(p)}{\partial p} \right| &= |\mathbb{I} + t\mathbb{K}(p)| \\ &= \prod_{i=1}^{n-1} (1 + \kappa_i(p) t) \\ &= \sum_{i=0}^{n-1} \mu_i^{n-1}(\kappa_1(p), \ldots, \kappa_{n-1}(p)) t^i. \end{aligned}$$

In other words, the determinant of the Jacobian is a polynomial in t whose coefficients are the elementary symmetric functions in the principal curvatures. With this substitution we can trivially integrate over the variable t and get

$$V(M_\epsilon) = V_n(M) + \sum_{i=0}^{n-1} \frac{\epsilon^{i+1}}{i+1} \int_{\partial M} \mu_i^{n-1}(\kappa_1(p), \ldots, \kappa_{n-1}(p))\, dp.$$

Comparing the two versions of Steiner's formula we see that the intrinsic volumes generalize curvature integrals. For example, for $n = 3$ and an arbitrary convex body K we get

$$V(K_\epsilon) = 1V_3(K) + 2V_2(K)\epsilon + \pi V_1(K)\epsilon^2 + \frac{4\pi}{3} V_0(K)\epsilon^3$$

while for a convex body M with C^2 smooth boundary the formula reads as

$$\begin{aligned} V(M_\epsilon) &= V_3(M) + \\ &\quad \underbrace{\left(\int_{\partial M} \underbrace{\mu_0^2(\kappa_1(p), \kappa_2(p))}_{=1} \, dp \right)}_{=A} \epsilon + \\ &\quad \left(\int_{\partial M} \underbrace{\mu_1^2(\kappa_1(p), \kappa_2(p))}_{=2H} \, dp \right) \frac{\epsilon^2}{2} + \\ &\quad \underbrace{\left(\int_{\partial M} \underbrace{\mu_2^2(\kappa_1(p), \kappa_2(p))}_{=K} \, dp \right)}_{=4\pi} \frac{\epsilon^3}{3} \\ &= V_3(M) + \epsilon \int_{\partial M} dp + \epsilon^2 \int_{\partial M} H\, dp + \frac{\epsilon^3}{3} \int_{\partial M} K\, dp. \end{aligned}$$

6 What All This Machinery Tells Us

We began this section by considering the question of what additive, continuous, rigid motion invariant measurements there are for convex bodies in $\mathbb{R}^n$ and learned that the $n + 1$ intrinsic volumes are the only ones and any such measure must be a linear combination of these. We have also seen that the intrinsic volumes in a natural way extend the idea of curvature integrals over the boundary of a smooth body to general convex bodies without regard to a differentiable structure. These considerations become one possible basis on which to claim that integrals of Gaussian curvature on a triangle mesh become sums over excess angle at vertices and that integrals of mean curvature can be identified with sums over edges of dihedral angle weighted by edge length. These quantities are always *integrals*. Consequently *they do not make sense as pointwise quantities*. In the case of smooth geometry we can define quantities such as mean and Gaussian curvature as pointwise quantities. On a simplicial mesh they are only defined as integral quantities.

All this machinery was developed for convex bodies. If a given mesh is not convex the additivity property allows us to compute the quantities anyway by writing the mesh as a finite union and intersection of convex bodies and then tracking the corresponding sums and differences of measures. For example, $V(K_\epsilon)$ is well defined for an individual triangle K and we know how to identify the coefficients involving intrinsic volumes with the integrals of elementary polynomials in the principal curvatures. Gluing two triangles together we can perform a similar identification carefully teasing apart the intrinsic volumes of the union of the two triangles. In this way the convexity requirement is relaxed so long as the shape of interest can be decomposed into a finite union of convex bodies.

This machinery was used by Cohen-Steiner and Morvan to give formulæfor integrals of a discrete curvature tensor. We give these here together with some fairly straightforward intuition regarding the underlying geometry.

Let P be a polyhedron with vertex set V and edge set E and B a small region (*e.g.*, a ball) in $\mathbb{R}^3$ then we can define integrated Gaussian and mean curvature measures as

$$\phi_P^K(B) = \sum_{v \in V \cap B} K_v \quad \text{and} \quad \phi_P^H(B) = \sum_{e \in E} l(e \cap B)\theta_e,$$

where $K_v = 2\pi - \sum_j \alpha_j$ is the excess angle sum at vertex v defined through all the incident triangle angles at v, while $l(\cdot)$ denotes the length and θ_e is the signed dihedral angle at e made between the incident triangle normals. Its sign is positive for convex edges and negative for concave edges (note that this requires an orientation on the polyhedron). In essence this is simply a restatement of the Steiner polynomial coefficients restricted to the intersection of the ball B and the polyhedron P. To talk about the second fundamental form II_p at some point p in the surface, it is convenient to first extend it to all of $\mathbb{R}^3$. This is done by setting it to zero if one of its arguments is parallel to the normal p. With this one may define

$$\overline{II}_P(B) = \sum_{e \in E} l(e \cap B)\theta_e e_n \otimes e_n, \quad e_n = e/\|e\|.$$

The dyad $(e_n \otimes e_n)(u, v) := \langle u, e_n \rangle \langle v, e_n \rangle$ projects given vectors u and v along the normalized edge.

What is the geometric interpretation of the summands? Consider a single edge and the associated dyad. The curvature along this edge is zero while it is θ orthogonal to the edge. A vector aligned with the edge is mapped to θ_e while one orthogonal to the edge is mapped to zero. These are the principal curvatures *except they are reversed*. Hence $\overline{II}_P(B)$ is an integral measure of the curvature tensor *with the principal curvature values exchanged*. For example we can assign each vertex a three-by-three matrix by summing the edge terms for each incident edge. As a tangent plane at the vertex, which we need to project the three-by-three matrix to the expected two-by-two matrix in the tangent plane, we may take a vector parallel to the area gradient at the vertex. Alternatively we could defined $\overline{II}_P(B)$ for balls containing a single triangle and its three edges each. In that case the natural choice for the tangent plane is the support plane of the triangle.

Cohen-Steiner and Morvan show that this definition can be rigorously derived from considering the coefficients of the Steiner polynomial in particular in the presence of nonconvexities (which requires some fancy footwork...). They also show that if the polyhedron is a sufficiently fine sample of a smooth surface the discrete curvature tensor integrals have linear precision with regards to continuous curvature tensor integrals. They also provide a formula for a discrete curvature tensor which does not have the principal curvatures swapped.

In practice one often finds that noise in the mesh vertex positions makes these discrete computations numerically delicate. One potential fix is to enlarge B to stabilize the computations. More in depth analyses of numerically reliable methods to estimate the curvature tensor have been undertaken by Yang *et al.* [Yang et al. 2006] and Grinspun *et al.* [Grinspun et al. 2006].

7 Further Reading

The material in this chapter only gives the rough outlines of what is a very fundamental theory in probability and geometric measure theory. In particular there are many other consequences which follow from relationships between intrinsic volumes which we have not touched upon. A rigorous derivation of the results of Hadwiger [Hadwiger 1957], but much shorter than the original can be found in [Klain 1995]. A complete and rigorous account of the derivation of intrinsic volumes from first principles in geometric probability can be found in the short book by Klain and Rota [Klain and Rota 1997], while the details of the discrete curvature tensor integrals can be found in [Cohen-Steiner and Morvan 2003]. Approximation results which discuss the accuracy of these measure vis-a-vis an underlying smooth surface are treated by Cohen-Steiner and Morvan in a series of tech reports available at http://www-sop.inria.fr/geometrica/publications/.

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